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# $\pi$-Blocks and Isometries

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In general, the "Third Main Theorem" does not hold for $\pi$-blocks. In this paper, however, we show that a "Third Main Theorem" does hold in certain cases and give applications to isometries.

We prove

THEOREM 1. *Let $G$ be a finite group, $\pi$ be a subset of $\pi(G)$. Let $L$ be a subgroup of $G$, $A$ be a union of $\pi$-sections of $L$. Assume:*

(i) *$G$ has a nilpotent Hall $\pi$-subgroup, $H$ say, and $N_G(X)$ is $\pi$-solvable whenever $1 \neq X \leqslant H$.*

(ii) *$L$ has a nilpotent Hall $\pi$-subgroup, and whenever two $\pi$-elements of $A$ are conjugate in $G$ they are conjugate in $L$.*

(iii) *For each $\pi$-element $a \in A$, $C_G(a) = C_L(a) O_{\pi'}(C_G(a))$.*

*Then there is an isometry $\sigma\colon V_0^\pi(L, A) \to V_0^\pi(G, A^*)$ such that $\theta^\sigma = \sum_{\chi \in B_0^\pi(G)} (\theta^G, \chi)\chi$ for any $\theta \in V_0^\pi(L, A)$ (where $V_0^\pi(L, A)$ is the space of class functions which are linear combinations of characters in the principal $\pi$-block of $L$, and which vanish outside $A$, and where $A^*$ is the smallest union of $\pi$-sections of $G$ containing $A$).*

*Remark.* In particular, $\sigma$ sends generalized characters to generalized characters. In the above, $B_0^{(\pi)}(G)$ denotes the $\pi$-block which contains the trivial character (the "principal" $\pi$-block).

We also prove

THEOREM 2. *Let $G$, $\pi$ be as above. Suppose that condition* (i) *of Theorem* 1 *is satisfied and that $\pi$ consists of odd primes. If $3 \in \pi$, suppose that*

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*$Qd(3)$ is not involved in $G$. Then the remaining hypotheses of Theorem 1 are satisfied with $L = N_G(ZJ(H))$, $A = \{\pi$-singular elements of $L\}$. Furthrmore, if $L/O_{\pi'}(L)$ is a Frobenius group with Frobenius kernel $HO_{\pi'}(L)/O_{\pi'}(L)$, all irreducible characters in the principal $\pi$-block of $G$ are constant on $\pi$-sections of non-identity $\pi$-elements ($J(H) = \langle J(O_p(H))$: $p \in \pi\rangle$, where $J(O_p(H))$ denotes the usual Thompson subgroup of $O_p(H)$ (i.e., the subgroup generated by its Abelian subgroups of maximal order)).*

Our first few results are true for arbitrary finite groups.

PROPOSITION 1. *Let $N$ be a normal $\pi'$-subgroup of the group $G$. Then*:

(i) $\mathrm{Irr}(G/N)$ *is a union of $\pi$-blocks of $G$.*

(ii) *For any $\pi$-element $x \in G$, $B_N(x)$ (the union of $\pi$-blocks of $C_G(x)$ corresponding to* $\mathrm{Irr}(G/N)$ *in Theorem* 6 *of* [4]) *is precisely* $\mathrm{Irr}(C_G(x)/C_N(x))$.

*Proof.* (i) $|G|^{-1} \sum_{\chi \in \mathrm{Irr}(G/N)} \chi(1) \sum_{g \in G} \chi(g^{-1})\, g = |N|^{-1} \sum_{n \in N} n$, which is an idempotent of $Z(R_\pi G)$, so $\mathrm{Irr}(G/N)$ is a union of $\pi$-blocks of $G$.

(ii) Let $x$ be a $\pi$-element of $G$, and let $y \in C_N(x)$. Then $\sum_{\chi \in \mathrm{Irr}(G/N)} \chi(x^{-1})\, \chi(xy) = \sum_{\mu \in B_N(x)} \mu(1)\, \mu(y)$ (as in Theorem 6 of [4]). On the other hand, since $y \in N$, we have $\chi(xy) = \chi(x)$ for all $\chi \in \mathrm{Irr}(G/N)$ so $\sum_{\mu \in B_N(x)} \mu(1)\, \mu(y) = \sum_{\chi \in \mathrm{Irr}(G/N)} |\chi(x)|^2 = |C_{G/N}(xN)| = |C_G(x)|/|C_N(x)|$ as $x$ is a $\pi$-element and $N$ is a normal $\pi'$-subgroup.

On the other hand, if $y \in C_G(x) \setminus C_N(x)$ and $y$ is $\pi$-regular then $xN$ and $xyN$ are not conjugate in $G/N$ as $xN$ is a $\pi$-element and $xyN$ is not. Hence $\sum_{\chi \in \mathrm{Irr}(G/N)} \chi(x^{-1})\, \chi(xy) = 0 = \sum_{\mu \in B_N(x)} \mu(1)\, \mu(y)$. Thus we have $\sum_{\mu \in B_N(x)} \mu(1) \sum_{y \in C_G(x)} \mu(y^{-1})\, y = \sum_{\beta \in \mathrm{Irr}(C_G(x)/C_N(x))} \beta(1) \sum_{y \in C_G(x)} \beta(y^{-1})\, y$, and the proof of Proposition 1 is complete.

PROPOSITION 2. *Let $G$ be a finite group such that $F^*(G)$ is a $\pi$-group. Then if $G$ contains a $\pi$-element such that $O_{\pi'}(C_G(x)) \neq 1$, $B_0(x)$ (the union of $\pi$-blocks of $C_G(x)$ corresponding to the principal $\pi$-block of $G$ as in Theorem $G$ of* [4]), *is* not *the principal $\pi$-block of $C_G(x)$.*

*Proof.* By Theorem 9 of [4], $G$ has only one $\pi$-block. Thus $B_0(x) = \mathrm{Irr}(C_G(x)$, whilst the principal $\pi$-block of $C_G(x)$ is certainly a subset of $\mathrm{Irr}(C_G(x)/O_{\pi'}(C_G(x)))$.

EXAMPLE. Let $G$ be a Frobenius group of order 42, $\pi = \{2, 7\}$. Then $O_{\pi'}(G) = 1_G$ so $F^*(G)$ is a $\pi$-group. Let $x$ be an involution of $G$. Then $O_{\pi'}(C_G(x))$ has order 3, so the "Third Main Theorem for $\pi$-blocks" fails within $G$.

For the remainder of this section, unless otherwise stated, we assume that $G$ is a finite group with a nilpotent Hall $\pi$-subgroup $H$ (where $\pi \subset \pi(G)$) that $C_G(X)$ is $\pi$-solvable whenever $1 \neq X \leqslant H$, and that $|\pi| \geqslant 2$. We will show that the "Third Main Theorem for $\pi$-blocks" holds within $G$ under these conditions. The assumption that $|\pi| \geqslant 2$ is no loss of generality, as Brauer's Third Main Theorem holds in the case $|\pi| = 1$ and is equivalent to the formulation we are using.

The first Lemma is very easy.

LEMMA 3. (i) *$N_G(X)$ is $\pi$-solvable whenever $1 \neq X \leqslant H$.*

(ii) *Whenever $R$, $S$ are non-trivial $\pi$-groups which centralize each other, $O_{\pi'}(C_G(R)) \cap C_G(S) \leqslant O_{\pi'}(C_G(S))$.*

*Proof.* (i) We may suppose that $X$ is a $p$-group for some prime $p \in \pi$. Let $T = O_{p'}(H)$. Then by a theorem of Wielandt [5], every Hall $(\pi \setminus \{p\})$-subgroup of $C_G(X)$ is conjugate to $T$, so $N_G(X) = C_G(X)(N_G(X) \cap N_G(T))$. By assumption $C_G(X)$ is $\pi$-solvable, so it suffices to show that $N_G(T)$ is $\pi$-solvable. However, $N_G(T) = C_G(T)\, N_G(H)$ since every Hall $\pi$-subgroup of $TC_G(T)$ is conjugate to $H$. Now $C_G(T)$ is $\pi$-solvable, and $N_G(H)$ is certainly $\pi$-solvable, so $N_G(T)$ is $\pi$-solvable, as required.

(ii) $RS$ is a $\pi$-group, so is conjugate to a subgroup of $H$ by the theorem of Wielandt mentioned above. We may as well assume, then, that $RS \leqslant H$. In particular, $C_G(S)$ is $\pi$-solvable. It suffices to prove that whenever $X$ is a $\pi$-solvable group with a nilpotent Hall $\pi$-subgroup and with $O_{\pi'}(X) = 1$, then $O_{\pi'}(C_X(T)) = 1$ for every $\pi$-subgroup, $T$, of $X$.

For such an $X$, $F^*(X)$ is a $\pi$-group, so $F^*(X) = F(X)$. It suffices to prove that $[F(X),.O_{\pi'}(C_X(T))] = 1$ whenever $T$ is a $\pi$-subgroup of $X$. Now $TF(X)$ is a $\pi$-group, so is nilpotent, so $[F(X), T; n] = 1$ for some $n$. Suppose that $O_{\pi'}(C_X(T))$ does not centralize $F(X)$. Then we can choose an integer $k > 0$ minimal subject to: $O_{\pi'}(C_X(T))$ centralizes $[F(X), T; k]$. Let $U = [F(X), T; k-1]$. Then $[U, T, O_{\pi'}(C_X(T))] = [T, O_{\pi'}(C_X(T)), U] = 1$, so $[O_{\pi'}(C_X(T)), U, T] = 1$. Thus $[O_{\pi'}(C_X(T)), U] \leqslant C_X(T)$, so normalizes $O_{\pi'}(C_X(T))$. On the other hand, $[O_{\pi'}(C_X(T)), U]$ is a $\pi$-group and is normalized by $O_{\pi'}(C_X(T))$, so $[O_{\pi'}(C_X(T)), U, O_{\pi'}(C_X(T))] = 1$. Thus $[O_{\pi'}(C_X(T)), U] = 1$ as $U$ is a $\pi$-group and $O_{\pi'}(C_X(T))$ is a $\pi'$-group normalizing $U$. This contradicts the choice of $k$, so that proof of Lemma 3 is complete.

PROPOSITION 4. *Assume that $\pi$ consists of odd primes and that $Qd(3)$ is not involved in $G$ if $3 \in \pi$. Let $L = N_G(ZJ(H))$. Then whenever $1 \neq X \leqslant H$, we have $N_G(X) = O_{\pi'}(C_G(X))\, N_L(X)$. Furthermore, whenever $X^g \leqslant L$ for some $g \in G$, we may write $g = cu$, where $c \in C_G(X)$, $u \in L$. In particular, any two $\pi$-elements of $H$ which are conjugate in $G$ are conjugate in $L$, and $C_G(x) = C_L(x)\, O_{\pi'}(C_G(x))$ whenever $x \in H^{\#}$.*

*Proof.* Once the first assertion is established, the result follows from Lemma 7 and Proposition E of [3].

If the first assertion is false, we may choose a subgroup $X \neq 1$ of $H$ with $|N_H(X)|$ maximal subject to $N_G(X) \neq O_{\pi'}(C_G(X))\, N_L(X)$. Let $T$ be a Hall $\pi$-subgroup of $N_L(X)$. Then $T^u \leqslant H$ for some $u \in L$. Then $N_H(X^u)$ is a Hall $\pi$-subgroup of $N_L(X^u)$ and $N_G(X^u) \neq O_{\pi'}(C_G(X^u))\, N_L(X^u)$, so by the choice of $X$, $N_H(X)$ is a Hall $\pi$-subgroup of $N_L(X)$.

We note that if $M$ is a $\sigma$-solvable group with $O_{\sigma'}(M) = 1$ and with a nilpotent Hall $\sigma$-subgroup $S$, then $ZJ(S) \lhd M$ if $\sigma$ is a set of odd primes and $Qd(3)$ is not involved in $M$ if $3 \in \sigma$. Choose $p \in \sigma$, and let $P \in \mathrm{Syl}_p(S)$. Then $M = O_{p'}(M)\, N_M(ZJ(P))$. Now $F^*(O_{p'}(M)) = F(O_{p'}(M))$ is a $\sigma$-group, so $P$ centralizes $F^*(O_{p'}(M))$ and hence also centralizes $O_{p'}(M)$. Thus $O_{p'}(M) \leqslant N_M(ZJ(P))$ so $ZJ(P) \lhd M$ as $p$ was arbitrary.

Let $R = N_H(X)$. Then $R > X$ (for certainly $X \neq H$). Now $N_G(R) = O_{\pi'}(C_G(R))\, N_L(R)$ by the choice of $X$ (for either $R = H$ or $N_H(R) > R$). Hence $N_G(R) \cap N_G(X) = O_{\pi'}(C_G(R))\ (N_L(R) \cap N_G(X))$. Thus $R$ is a Hall $\pi$-subgroup of $N_G(R) \cap N_G(X)$, so is a Hall $\pi$-subgroup of $N_G(X)$ (as $N_G(X)$ has a nilpotent Hall $\pi$-subgroup). By the argument above, $N_G(X) = O_{\pi'}(C_G(X))\ (N_G(ZJ(R)) \cap N_G(X))$. Furthermore, $N_G(ZJ(R)) = O_{\pi'}(N_G(ZJ(R)))\, N_L(ZJ(R))$ by the choice of $X$. By Lemma 6 of [3], $N_G(ZJ(R)) \cap N_G(X) = O_{\pi'}(N_G(ZJ(R) \cap N_G(X))\, (N_L(ZJ(R)) \cap N_G(X))$. Since $N_G(X) = O_{\pi'}(C_G(X))\ (N_G(X) \cap N_G(ZJ(R))$, we have $O_{\pi'}(N_G(X) \cap N_G(ZJ(R))) \leqslant O_{\pi'}(C_G(X))$. Consequently, $N_G(X) = O_{\pi'}(C_G(X))\, N_L(X)$, contrary to the choice of $X$. The proof of Proposition 4 is complete.

LEMMA 5. *Let $M$ be a $\tau$-solvable group with a nilpotent Hall $\tau$-subgroup for some set of primes $\tau$, and with $O_{\tau'}(M) = 1$. Then for any prime $p \in \tau$, each $p$-block of $M$ is a block of full defect.*

*Proof.* Let $X \in \mathrm{Syl}_p(O_{p',p}(M))$. Then $M = O_{p'}(M)\, N_M(X)$. Now $F^*(O_{p'}(M))$ is a $\tau$-group, so $F^*(O_{p'}(M)) = F(O_{p'}(M))$, and $[X, F(O_{p'}(M))] = 1$. It readily follows that $[X, O_{p'}(M)] = 1$, so $X \lhd M$.

Let $D$ be a defect group for some $p$-block of $M$. Then $X \leqslant D$. Let $P \in \mathrm{Syl}_p(M)$ with $D \leqslant P$. Then for some $p$-regular $y \in M$, $D \in \mathrm{Syl}_p(C_M(y))$. Since $M$ is $p$-solvable and $y$ centralizes a Sylow $p$-subgroup of $O_{p',p}(M)$, $y$ lies in $O_{p'}(M)$. Now $[P, F^*(O_{p'}(M))] = 1$, so $[P, O_{p'}(M)] = 1$, so $[P, y] = 1$, and $D = P$, as required.

THEOREM 6. *Whenever $x \in H^{\#}$, $B_0(x)$ (the union of $\pi$-blocks of $C_G(x)$ corresponding to the principal $\pi$-block of $G$ in Theorem 6 of [4]) is the principal $\pi$-block of $C_G(x)$.*

*Proof.* By Lemma 3 and the results of Puig [1, Sect. 2], for each prime $p \in \pi$, there is a union of $p$-blocks of $G$, say $B_p^{(\pi)}$, such that for each $p$-

element $x \in G$, $\sum_{\chi \in B_p(\pi)} \chi(x^{-1}y^{-1})\,\chi(xz) = \sum_{\mu \in \mathrm{Irr}(C_G(x)/O_{\pi'}(C_G(x)))} \mu(y^{-1})\,\mu(z)$ whenever $y, z \in C_G(x)_{p'}$ (translating into the terms of Theorem 6 of [4]). We may, and do, assume that $B_p^{(\pi)}$ consists of $p$-blocks of positive defect. We will prove that in this case $B_p^{(\pi)} = B_q^{(\pi)}$ whenever $p, q \in \pi$.

We first prove that $B_p^{(\pi)}$ is a union of $p$-blocks of full defect. Let $D$ be a defect group for a $p$-block $B \subset B_p^{(\pi)}$. Then $D$ is the defect group of a $p$-block of $N_G(D)$ associated with an idempotent occurring in the decomposition of $\mathrm{Br}_p(e_p^{(\pi)})$, where $e_p^{(\pi)}$ is the idempotent of $Z(FG)$ associated with $B_p^{(\pi)}$ ($F$ a splitting field of characteristic $p$), and where $\mathrm{Br}_p$ is the Brauer homomorphism from $Z(FG)$ to $Z(FN_G(D))$.

By the results of Puig [1, Sect. 2], and Lemma 3, $\mathrm{Br}_p(e_p^{(\pi)}) = |O_{\pi'}(C_G(D))|^{-1} \sum_{u \in O_{\pi'}(C_G(D))} u$ (in $Z(FC_G(D))$). Thus any block idempotent occurring in the decomposition of $\mathrm{Br}_p(e_p^{(\pi)})$ is "really" a block idempotent of $N_G(D)/O_{\pi'}(C_G(D))$. By Lemma 5, any $p$-block of $N_G(D)/O_{\pi'}(C_G(D)$ has full defect. Thus $D \in \mathrm{Syl}_p(N_G(D))$, so $D \in \mathrm{Syl}_p(G)$, as required to establish our claim.

Now choose a prime $p \in \pi$ and a $p$-singular $\pi$-element $x$ with $p$ part $x_p$. Then whenever $y, z \in C_G(x_p)_{p'}$

$$\sum_{\chi \in B_p^{(\pi)}} \chi(x_p^{-1}y^{-1})\,\chi(x_p z) = \sum_{\mu \in \mathrm{Irr}(C_G(x_p)/O_{\pi'}(C_G(x_p)))} \mu(y^{-1})\,\mu(z).$$

By Lemma 3, $O_{\pi'}(C_G(x)) = O_{\pi'}(C_G(x_p)) \cap C_G(x)$. By Proposition 1, we have (whenever $u, v \in C_G(x)_{\pi'}$)

$$\sum_{\mu \in \mathrm{Irr}(C_G(x_p)/O_{\pi'}(C_G(x_p)))} \mu(x_{p'}^{-1}u^{-1})\,\mu(x_{p'}v) = \sum_{\beta \in \mathrm{Irr}(C_G(x)/O_{\pi'}(C_G(x)))} \beta(u^{-1})\,\beta(v).$$

Thus whenever $u, v \in C_G(x)_{\pi'}$ we have

$$\begin{aligned}\sum_{\chi \in B_p^{(\pi)}} \chi(x^{-1}u^{-1})\,\chi(xv) &= \sum_{\mu \in \mathrm{Irr}(C_G(x_p)/O_{\pi'}(C_G(x_p)))} \mu(x_{p'}^{-1}u^{-1})\,\mu(x_{p'}v)\\ &= \sum_{\beta \in \mathrm{Irr}(C_G(x)/O_{\pi'}(C_G(x)))} \beta(u^{-1})\,\beta(v).\end{aligned}$$

By Lemma 4 of [4] for any $\chi \in B_p^{(\pi)}$ and any $y \in C_G(x)_{\pi'}$, we have $\chi(y) = \sum_{\beta \in \mathrm{Irr}(C_G(x)/O_{\pi'}(C_G(x)))} (\chi|_{C_G(x)}, \beta)\,\beta(y)$ (in particular, $\chi(xyz) = \chi(xy)$ whenever $z \in O_{\pi'}(C_G(x))$), and furthermore whenever $\chi' \in \mathrm{Irr}(G)$ is such that there is a fixed set of complex numbers $\{c_\beta : \beta \in \mathrm{Irr}(C_G(x)/O_{\pi'}(C_G(x))\}$ such that $\chi'(xy) = \sum_{\beta \in \mathrm{Irr}(C_G(x)/O_{\pi'}(C_G(x)))} c_\beta\,\beta(xy)$ for all $y \in C_G(x)_{\pi'}$, then either $\chi' \in B_p^{(\pi)}$, or $\chi'$ vanishes on the $\pi$-section of $x$ in $G$.

Suppose that for some $\chi' \in \mathrm{Irr}(G)$ we have $\chi'(xyz) = \chi'(xy)$ whenever $y \in C_G(x)_{\pi'}$ and $z \in O_{\pi'}(C_G(x))$. Let $\beta \in \mathrm{Irr}(C_G(x))$ with $O_{\pi'}(C_G(x)) \nleq \ker \beta$. Then for $y \in C_G(x)_{\pi'}$, $\sum_{z \in O_{\pi'}(C_G(x))} \chi'(xyz)\,\overline{\beta(xyz)} = \chi'(xy) \sum_{z \in O_{\pi'}(C_G(x))} \overline{\beta(xyz)}$

$=0$. Since $C_G(x)_{\pi'}$ is a union of cosets of $O_{\pi'}(C_G(x))$, it readily follows that we may write (for each $y \in C_G(x)_{\pi'}$) $\chi'(xy) = \sum_{\beta \in \mathrm{Irr}(C_G(x)/O_\pi(C_G(x))} c_\beta \beta(xy)$, where the $c_\beta$'s are fixed complex numbers. Thus either $\chi' \in B_p^{(\pi)}$ or else $\chi'$ vanishes identically on the $\pi$-section of $x$.

Hence we can characterize $B_p^{(\pi)}$ as follows:

(i) For each $\chi \in B_p^{(\pi)}$, each $p$-singular $\pi$-element, $x$, of $G$, each $y \in C_G(x)_{\pi'}$ and each $z \in O_{\pi'}(C_G(x))$ we have $\chi(xyz) = \chi(xy)$.

(ii) Suppose that $\chi$ is an irreducible character of $G$ such that there is a $p$-singular $\pi$-element, $x$, of $G$ such that $\chi$ does not vanish on $S_\pi^G(x)$ and $\chi(xyz) = \chi(xy)$ whenever $y \in C_G(x)_{\pi'}$ and $z \in O_{\pi'}(C_G(x))$. Then $\chi \in B_p^{(\pi)}$.

We want to prove that $B_p^{(\pi)} = B_q^{(\pi)}$ whenever $p, q \in \pi$. Let $p, q \in \pi$, $(p \neq q)$, $P \in \mathrm{Syl}_p(H)$, $Q \in \mathrm{Syl}_q(H)$. It suffices to prove that $B_p^{(\pi)} \subset B_q^{(\pi)}$. If possible, choose $\chi \in B_p^{(\pi)} \setminus B_q^{(\pi)}$. Choose $a \in Z(P)^\#$, $b \in Q^\#$, and let $x = ab$. Suppose that $\chi(x) \neq 0$. Then whenever $y \in C_G(x)_{\pi'}$ and $z \in O_{\pi'}(C_G(x))$ we have $\chi(xyz) = \chi(xy)$, since $x$ is a $p$-singular $\pi$-element and $\chi \in B_p^{(\pi)}$. Since $x$ is also a $q$-singular $\pi$-element, we have $\chi \in B_q^{(\pi)}$ by the remarks above, contrary to assumption. Thus whenever $a \in Z(P)^\#$ and $b \in Q^\#$, $\chi(ab) = 0$.

Let $N = N_G(P)$. Then any two elements of $C_G(P)$ which are conjugate in $G$ are already conjugate in $N$. We know that $\chi$ lies in a $p$-block with defect group $P$, and that the corresponding block of $N$ has $O_{\pi'}(N)$ in its kernel. Fix $a \in Z(P)^\#$. There is an irreducible character $\mu$ of $N/O_{\pi'}(N)P$ such that for all $b \in Q^\#$ we have $[G: C_G(ab)]\, \chi(ab)/\chi(1) \equiv [N: C_N(ab)]\, \mu(ab)/\mu(1)$ (mod $\rho$), where $\rho$ is a prime ideal of $\mathbb{Z}[\exp(2\pi i/|G|)]$ containing $p$.

Since $\chi(ab) = 0$ for each $b \in Q^\#$ we see that $[N: C_N(ab)]\, \mu(ab)/\mu(1) \in \rho$ for each $b \in Q^\#$, so $\mu(b) \in \rho$ for each $b \in Q^\#$ as $P \leqslant \ker \mu$ and $p \nmid [N: C_N(ab)]$.

Let $\bar{N} = N/O_{\pi'}(N)$. Then $\bar{N}$ is $\pi$-solvable with nilpotent Hall $\pi$-subgroup $\bar{H}$. Now $\bar{Q}$ centralizes $F(O_{q'}(\bar{N})) = F^*(O_{q'}(\bar{N}))$, so $\bar{Q}$ centralizes $O_{q'}(\bar{N})$. Thus $O_q(\bar{N}) \in \mathrm{Syl}_q(O_{q',q}(\bar{N}))$, and in particular, $O_q(\bar{N}) \neq \bar{1}$.

Let $R$ be the pre-image in $Q$ of $O_q(\bar{N})$. For any $b \in R^\#$, $\mu(b) \in \rho$. Now $\bar{R} \lhd \bar{N}$, and $\sum_{b \in \bar{R}} \mu(\bar{b}) \equiv \mu(1)$ (mod $\rho$), so $(\mu|_{\bar{R}}, 1) \neq 0$ since $p \nmid \mu(1)$. Since $\bar{R} \lhd \bar{N}$, we must have $\bar{R} \leqslant \ker \mu$. Then for any $b \in R$ we have $\mu(b) = \mu(1) \notin \rho$ contrary to the fact that $\mu(b) \in \rho$ for all such $b$. This contradiction shows that $B_p^{(\pi)} = B_q^{(\pi)}$ whenever $p, q \in \pi$.

It now follows that $B_p^{(\pi)}$ is a union of $\pi$-blocks of $G$. Now for any $\pi$-element $x \in G^\#$, $B_0^{(\pi)}(C_G(x)) = \mathrm{Irr}(C_G(x)/O_{\pi'}(C_G(x)))$ by Theorem 9 of [4] and by Proposition 1 (since $F^*(C_G(x)/O_{\pi'}(C_G(x)))$ is a $\pi$-group). It is clear that $1 \in B_p^{(\pi)}$ for each $p \in \pi$, and hence that $B_0^{(\pi)}(G) \subset B_p^{(\pi)}$ Thus whenever $x \in G^\#$ is a $\pi$-element we have $\sum_{\chi B_0^{(\pi)}(G)} |\chi(x)|^2 \leqslant \sum_{\mu \in B_0^{(\pi)}(C_G(x))} |\mu(x)|^2$ whilst also from Theorem 6 of [4], $\sum_{\chi \in B_0^{(\pi)}(G)} |\chi(x)|^2 = \sum_{\beta \in B(x)} |\beta(x)|^2$ for some union of $\pi$-blocks, $B(x)$ say, of $C_G(x)$ and from Lemma 4 of [4] it also follows that for each $y \in C_G(x)_{\pi'}$ we have $1 = \sum_{\beta \in B(x)} (\beta, 1)\, \beta(xy)$, so $1 \in B(x)$ and $B_0^{(\pi)}(C_G(x)) \subset B(x)$. Thus

$$\sum_{\chi \in B_0^{(\pi)}(G)} |\chi(x)|^2 \geqslant \sum_{\mu \in B_0^{(\pi)}(C_G(x))} |\mu(x)|^2.$$

Hence $\chi(x) = 0$ whenever $\chi \in B_p^{(\pi)} \backslash B_0^{(\pi)}(G)$. Since this applies to each $x \in H^{\#}$, $|H| \mid \chi(1)$ for each such $\chi$ (as $(\chi|_H, 1_H) = \chi(1)/|H|$), contrary to the fact that $B_p^{(\pi)}$ consists of $p$-blocks of positive defect for each prime $p \in \pi$. Hence $B_p^{(\pi)} = B_0^{(\pi)}(G)$, as required.

*Proof of Theorem* 2. It only remains to prove that when $L/O_{\pi'}(L)$ is a Frobenius group with kernel $HO_{\pi'}(L)/O_{\pi'}(L)$, all irreducible characters in the principal $\pi$-block of $G$ are constant on $\pi$-sections of non-identity $\pi$-elements. Let $x$ be a $\pi$-element of $G^{\#}$. Then some conjugate of $x$ lies in $H$, so we may assume that $x \in H$, and we do so. Then $C_G(x) = C_L(x)\, O_{\pi'}(C_G(x))$ by Proposition 4. Now $C_L(x)/(O_{\pi'}(L) \cap C_L(x))$ is a $\pi$-group by assumption, so $C_L(x)$ has a normal $\pi$-complement, and $C_G(x)$ does also. For any $\chi \in B_0^{(\pi)}(G)$ and any $y \in C_G(x)_{\pi'}$ we have $\chi(xy) = \chi(x)$ since $y \in O_{\pi'}(C_G(x))$ and $\chi \in B_q^{(\pi)}$ for any prime $q \in \pi$ with $q \mid |\langle x \rangle|$.

*Proof of Theorem* 1. We claim that $V_0^{\pi}(G, A^*)$ is the space of class functions, $\theta$, of $G$ which vanish outside $A^*$ and which satisfy $\theta(abc) = \theta(ab)$ whenever $a \in A^*$ is a $\pi$-element $b \in C_G(a)_{\pi'}$ and $c \in O_{\pi'}(C_G(a))$. It is clear from earlier remarks that $V_0^{(\pi)}(G, A^*)$ is contained in this space. On the other hand, it is easy to see that for $\theta$ as above and any $\pi$-element $a \in A$ such that $\theta$ does not vanish on $S_\pi^G(a)$, we can find complex numbers $c_\mu (\mu \in \mathrm{Irr}(C_G(a)/O_{\pi'}(C_G(a))))$ with $\theta(b) = \sum_{\mu \in \mathrm{Irr}(C_G(a) \cdot O_{\pi'}(C_G(a)))} c_\mu \mu(ab)$ for all $\pi$-regular $b \in C_G(a)_{\pi'}$. It easily follows from this that whenever $\chi \notin B_0^{(\pi)}(G)$ we have $\sum_{u \in S_\pi^G(a)} \theta(u)\, \overline{\chi(u)} = 0$. Hence $\theta \in V_0^{(\pi)}(G, A^*)$.

A similar characterization of $V_0^{(\pi)}(L, A)$ can be given. It follows from the results of [3] (for example) that $\sigma$ is an isometry. Since we have established a "Third Main Theorem" for $\pi$-blocks within $G$ (and by a similar argument, within $L$), the proof that $\theta^\sigma = \sum_{\chi \in B_0^{(\pi)}(G)} (\theta^G, \chi)\chi$ proceeds exactly as in Reynolds' proof in the single prime case in [2].

## References


1. L. Puig, Structure locale et charactères, *J. Algebra* **56** (1979), 24–42.
2. W. F. Reynolds, Isometries and principal blocks of group characters, *Math. Z.* **107** (1968), 264–270.
3. G. R. Robinson, Blocks, isometries and sets of primes, *Proc. London Math. Soc. (3)* **51** (1985), 432–448.
4. G. R. Robinson, Group algebras over semi-local rings, *J. Algebra* **117** (1988), 409–418.
5. H. Wielandt, Zum Satz von Sylow, *Math. Z.* **60** (1954), 407–408.